NOTES ON SEMICIRCULANTS.

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1. Some attention has been given to the eight-line determinant

$$\begin{vmatrix} a & b & c & d & e & f & g & h \\ h & a & b & c & d & e & f & g \\ g & h & a & b & c & d & e & f \\ f & g & h & a & b & c & d & e \\ h & g & f & e & d & c & b & a \\ a & h & g & f & e & d & c & b \\ b & a & h & g & f & e & d & c \\ c & b & a & h & g & f & e & d \end{vmatrix}$$

but apparently with no satisfactory result. The evaluation of it was set as a problem in the *Educational Times* for April, 1904, and a supposed solution was given in the number for September of the same year,* the result being that the value was found to be the same as that of the ordinary circulant $C(a, b, c, d, e, f, g, h)$. But although the first four rows of the given determinant are the same as those of the said circulant, the last four rows are really the first four rows of the reverse circulant $C(h, g, f, e, d, c, b, a)$. The matter thus seems to call for more careful investigation.

2. Let us first make a fresh examination of the ordinary circulant. Performing on it the operations

$$\text{col}_1 + \text{col}_3 + \text{col}_5 + \text{col}_7,$$
$$\text{col}_2 + \text{col}_4 + \text{col}_6 + \text{col}_8,$$

we find it equal to

$$\begin{vmatrix} \omega & \varepsilon & c & d & e & f & g & h \\ \varepsilon & \omega & b & c & d & e & f & g \\ \omega & \varepsilon & a & b & c & d & e & f \\ \varepsilon & \omega & h & a & b & c & d & e \\ \omega & \varepsilon & g & h & a & b & c & d \\ \varepsilon & \omega & f & g & h & a & b & c \\ \omega & \varepsilon & e & f & g & h & a & b \\ \varepsilon & \omega & d & e & f & g & h & a \end{vmatrix}$$

* See also *Math. from Educ. Times*, (2) vii., p. 55.

where ω denotes the sum of the odd-placed elements, and ε the sum of the others. If on this the second set of operations

$$\text{row}_8 - \text{row}_6, \quad \text{row}_7 - \text{row}_5, \quad \text{row}_6 - \text{row}_4, \quad \ldots\ldots$$

be performed, we have

$$\begin{vmatrix} \omega & \varepsilon & c & d & e & f & g & h \\ \varepsilon & \omega & b & c & d & e & f & g \\ . & . & a-c & b-d & c-e & d-f & e-g & f-h \\ . & . & h-b & a-c & b-d & c-e & d-f & e-g \\ . & . & g-a & h-b & a-c & b-d & c-e & d-f \\ . & . & f-h & g-a & h-b & a-c & b-d & c-e \\ . & . & e-g & f-h & g-a & h-b & a-c & b-d \\ . & . & d-f & e-g & f-h & g-a & h-b & a-c \end{vmatrix}$$

or

$$(\omega+\varepsilon)\,(\omega-\varepsilon)\,\Delta_6,$$

where Δ_6 is a function of the eight differences

$$a-c,\ b-d,\ c-e,\ d-f,\ e-g,\ f-h,\ g-a,\ h-b,$$

being in fact, save as to sign, the persymmetric determinant

$$\mathrm{P}\begin{pmatrix} & e-g & & g-a & & a-c & & c-e & & e-g & \\ d-f & & f-h & & h-b & & b-d & & d-f & & f-h \end{pmatrix}. \qquad \text{(I.)}$$

A similar result is of course obtainable in reference to all circulants of even order.

3. Taking now the given semicirculant, if we may so term it, and proceeding in almost quite the same manner, we find it equal to

$$\begin{vmatrix} \omega & \varepsilon & c & d & e & f & g & h \\ \varepsilon & \omega & b & c & d & e & f & g \\ . & . & a-c & b-d & c-e & d-f & c-g & b-h \\ . & . & h-b & a-c & b-d & c-e & b-f & a-g \\ . & . & f-h & e-a & d-b & . & . & . \\ . & . & g-a & f-b & e-c & . & . & . \\ . & . & h-b & g-c & f-d & . & . & . \\ . & . & a-c & h-d & g-e & . & . & . \end{vmatrix},$$

and therefore equal to 0, since the co-factor of $\omega^2-\varepsilon^2$ here vanishes.

Of course the same final result might be reached without removing the factor $\omega^2-\varepsilon^2$. Further, by performing in succession the operations

$$\text{row}_8 - \text{row}_1, \quad \text{row}_7 - \text{row}_2, \quad \text{row}_6 - \text{row}_3, \quad \ldots\ldots$$
$$\text{col}_7 + \text{col}_5, \quad \text{col}_8 + \text{col}_4, \quad \text{col}_1 + \text{col}_3, \quad \ldots\ldots,$$

we can make it appear that

$$\begin{vmatrix} g-a & f-b & e-c \\ h-b & g-c & f-d \\ a-c & h-d & g-e \end{vmatrix}$$

also is a factor, and that the vanishing co-factor of the 5th order may have $\omega+\epsilon$, $\omega-\epsilon$ removed from it, leaving only

$$\begin{vmatrix} g-a & f-b & e-c \\ h-b & g-c & f-d \\ g-a & f-b & e-c \end{vmatrix}$$

which differs from the previous three-line factor in having its third row identical with its first.

By any of the methods here indicated it is easy to establish the general theorem: *Every semicirculant of order* $4m$ *vanishes.* (II.)

4. In the case of the other even orders, say the order $4m+2$, the result is different. As before we readily see that $\omega+\epsilon$ and $\omega-\epsilon$ are factors: their co-factor, however, does not now vanish. Thus taking the six-line determinant

$$\begin{vmatrix} a & b & c & d & e & f \\ f & a & b & c & d & e \\ e & f & a & b & c & d \\ f & e & d & c & b & a \\ a & f & e & d & c & b \\ b & a & f & e & d & c \end{vmatrix}$$

and performing the operations

$$\mathrm{col}_1 + \mathrm{col}_3 + \mathrm{col}_5, \quad \mathrm{col}_6 + \mathrm{col}_4 + \mathrm{col}_2,$$
$$\mathrm{row}_6 - \mathrm{row}_2, \quad \mathrm{row}_5 - \mathrm{row}_3, \quad \mathrm{row}_4 - \mathrm{row}_2, \quad \mathrm{row}_3 - \mathrm{row}_1,$$

we find it

$$= \begin{vmatrix} \omega & b & c & d & e & \epsilon \\ \epsilon & a & b & c & d & \omega \\ . & f-b & a-c & b-d & c-e & . \\ . & e-a & d-b & . & b-d & . \\ . & . & e-a & d-b & . & . \\ . & . & f-b & e-c & . & . \end{vmatrix},$$

$$= \begin{vmatrix} e-a & d-b \\ f-b & e-c \end{vmatrix} \cdot \begin{vmatrix} f-b & c-e \\ e-a & b-d \end{vmatrix} \cdot \begin{vmatrix} \omega & \epsilon \\ \epsilon & \omega \end{vmatrix},$$

$$= (a+b+c+\dots)(a-b+c-d+\dots) \begin{vmatrix} e-a & f-b \\ d-b & e-c \end{vmatrix}^2.$$

5. The ten-line and other cases may be similarly dealt with. There is, however, a much more instructive method; for, if the last $m+1$ columns of such a determinant be moved in a body so as to occupy the first $m+1$ places, the determinant will be found to be centrosymmetric, and therefore resolvable into two determinants of the order $2m+1$, from one of which the factor $a+b+c+d+\ldots$ can be removed, and from the other the factor $a-b+c-d+\ldots$, leaving co-factors which are identical. Thus, shifting as stated the last three columns of the semicirculant of the 10th order, and using 1, 2, 3, ... for elements, we have it

$$= \begin{vmatrix} 8 & 9 & t & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 7 & 8 & 9 & t & 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 7 & 8 & 9 & t & 1 & 2 & 3 & 4 & 5 \\ 5 & 6 & 7 & 8 & 9 & t & 1 & 2 & 3 & 4 \\ 4 & 5 & 6 & 7 & 8 & 9 & t & 1 & 2 & 3 \\ 3 & 2 & 1 & t & 9 & 8 & 7 & 6 & 5 & 4 \\ 4 & 3 & 2 & 1 & t & 9 & 8 & 7 & 6 & 5 \\ 5 & 4 & 3 & 2 & 1 & t & 9 & 8 & 7 & 6 \\ 6 & 5 & 4 & 3 & 2 & 1 & t & 9 & 8 & 7 \\ 7 & 6 & 5 & 4 & 3 & 2 & 1 & t & 9 & 8 \end{vmatrix}$$

$$= \begin{vmatrix} 8+7 & 9+6 & t+5 & 1+4 & 2+3 \\ 7+6 & 8+5 & 9+4 & t+3 & 1+2 \\ 6+5 & 7+4 & 8+3 & 9+2 & t+1 \\ 5+4 & 6+3 & 7+2 & 8+1 & 9+t \\ 4+3 & 5+2 & 6+1 & 7+t & 8+9 \end{vmatrix} \cdot \begin{vmatrix} 8-7 & 9-6 & t-5 & 1-4 & 2-3 \\ 7-6 & 8-5 & 9-4 & t-3 & 1-2 \\ 6-5 & 7-4 & 8-3 & 9-2 & t-1 \\ 5-4 & 6-3 & 7-2 & 8-1 & 9-t \\ 4-3 & 5-2 & 6-1 & 7-t & 8-9 \end{vmatrix}.$$

The first of these factors, when there is performed on it the operations

$$\text{col}_1 + \text{col}_2 + \text{col}_3 + \ldots\ldots$$
$$\text{row}_5 - \text{row}_4, \quad \text{row}_4 - \text{row}_3, \quad \ldots\ldots$$

is seen to be

$$= (1+2+\ldots+t) \cdot \begin{vmatrix} 8+5-9-6 & 9+4-t-5 & t+3-1-4 & 1-3 \\ 7+4-8-5 & 8+3-9-4 & 9+2-t-3 & t-2 \\ 6+3-7-4 & 7+2-8-3 & 8+1-9-2 & 9-1 \\ 5+2-6-3 & 6+1-7-2 & 7+t-8-1 & 8-t \end{vmatrix},$$

$$= (1+2+\ldots+t) \cdot \begin{vmatrix} 8-6 & 9-5 & t-4 & 1-3 \\ 7-5 & 8-4 & 9-3 & t-2 \\ 6-4 & 7-3 & 8-2 & 9-1 \\ 5-3 & 6-2 & 7-1 & 8-t \end{vmatrix};$$

and similarly the second factor is found

$$= (1-2+3-\ldots-t) \begin{vmatrix} 8-6 & 9-5 & t-4 & 1-3 \\ 7-5 & 8-4 & 9-3 & t-2 \\ 6-4 & 7-3 & 8-2 & 9-1 \\ 5-3 & 6-2 & 7-1 & 8-t \end{vmatrix}.$$

As the common four-line factor is the determinant of the difference of the matrices

$$\begin{matrix} 8 & 9 & t & 1 \\ 7 & 8 & 9 & t \\ 6 & 7 & 8 & 9 \\ 5 & 6 & 7 & 8, \end{matrix} \qquad \begin{matrix} 6 & 5 & 4 & 3 \\ 5 & 4 & 3 & 2 \\ 4 & 3 & 2 & 1 \\ 3 & 2 & 1 & t \end{matrix}$$

we may say that the semicirculant of the ten elements 1, 2, 3, ..., t is equal to

$$(1+2+3+4+\ldots+t).(1-2+3-4+\ldots-t)$$
$$\cdot\begin{vmatrix} 8 & 9 & t & 1 & & 6 & 5 & 4 & 3 \\ 7 & 8 & 9 & t & & 5 & 4 & 3 & 2 \\ 6 & 7 & 8 & 9 & - & 4 & 3 & 2 & 1 \\ 5 & 6 & 7 & 8 & & 3 & 2 & 1 & t \end{vmatrix}^2.$$

6. The matrices whose difference is the matrix of the squared factor in the preceding are both persymmetric in form, and are both obtainable from the first $2m$ rows of the given semicirculant, after it has been made centrosymmetric, by leaving out the $(2m+1)$th and $(4m+2)$th columns, and in the case of the second matrix reversing the order of the columns. Thus in the case of the fourteen-line semicirculant (*i.e.*, where $m = 3$), if the first row be

$$1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 7,\ a,\ b,\ c,\ d,\ e,\ f,\ g$$

we shift the last four elements to the front, and so alter the row into

$$d,\ e,\ f,\ g,\ 1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 7,\ a,\ b,\ c;$$

then we leave out the 7th and 14th, obtaining

$$d,\ e,\ f,\ g,\ 1,\ 2 \qquad 4,\ 5,\ 6,\ 7,\ a,\ b,$$

and thus finally are led to the squared factor

$$\begin{vmatrix} d & e & f & g & 1 & 2 & & b & a & 7 & 6 & 5 & 4 \\ c & d & e & f & g & 1 & & a & 7 & 6 & 5 & 4 & 3 \\ b & c & d & e & f & g & - & 7 & 6 & 5 & 4 & 3 & 2 \\ a & b & c & d & e & f & & 6 & 5 & 4 & 3 & 2 & 1 \\ 7 & a & b & c & d & e & & 5 & 4 & 3 & 2 & 1 & g \\ 6 & 7 & a & b & c & d & & 4 & 3 & 2 & 1 & g & f \end{vmatrix}^2.$$

If, therefore, in addition to the usual notation for persymmetric determinants, viz.,

$$\mathrm{P}(a, b, c, d, e) \quad \text{for} \quad \begin{vmatrix} a & b & c \\ b & c & d \\ c & d & e \end{vmatrix}$$

we introduce

$$\mathrm{P}'(a, b, c, d, e) \quad \text{for} \quad \begin{vmatrix} c & d & e \\ b & c & d \\ a & b & c \end{vmatrix},$$

that is to say, for the determinant which is got from the former by changing the order of the rows and which therefore is symmetric with respect to the second diagonal instead of the first, we may enunciate our result as follows: *The semicirculant of the* $(4m+2)$*th order whose elements are* $1, 2, \ldots, 4m+2$ *is equal to* $(-)^{m-1}(\omega+\epsilon)(\omega-\epsilon)\Delta^2$, *where* ω, ϵ *are the sums of the odd-placed and even-placed elements respectively, and where* Δ *is the determinant of the difference between the matrices of the two persymmetric determinants* *

$$\begin{aligned} &\mathrm{P}'(m+3, m+4, \ldots, 4m+2, 1, 2, \ldots, m-1), \\ &\mathrm{P}(3m, 3m-1, \ldots, 1, 4m+2, 4m+1, \ldots, 3m+4). \end{aligned} \qquad \text{(III.)}$$

In connection with this it is interesting to note that in every case there are three elements not included in the first persymmetric matrix, viz., m, $m+1$, $m+2$, and likewise three not included in the second, viz., $3m+3$, $3m+2$, $3m+1$: further, that the element which fills the univarial diagonal in the first matrix is $3m+2$, and in the second matrix $m+1$.

7. If instead of having, as in the foregoing semicirculants, both the cyclical changes right-handed, we make one the opposite of the other, the resulting determinant will be found still resolvable into factors. In this case, however, no distinction is necessary between the orders $4m$ and $4m+2$, centrosymmetry being now always attainable by merely reversing the order of the last m of the $2m$ rows. Thus, when $m=3$ we have

$$\begin{vmatrix} a & b & c & d & e & f \\ f & a & b & c & d & e \\ e & f & a & b & c & d \\ f & e & d & c & b & a \\ e & d & c & b & a & f \\ d & c & b & a & f & e \end{vmatrix} = -\begin{vmatrix} a & b & c & d & e & f \\ f & a & b & c & d & e \\ e & f & a & b & c & d \\ d & c & b & a & f & e \\ e & d & c & b & a & f \\ f & e & d & c & b & a \end{vmatrix},$$

$$= -\begin{vmatrix} a+f & b+e & c+d \\ f+e & a+d & b+c \\ e+d & f+c & a+b \end{vmatrix} \cdot \begin{vmatrix} a-f & b-e & c-d \\ f-e & a-d & b-c \\ e-d & f-c & a-b \end{vmatrix},$$

$$= -(a+b+c+d+e+f)(a-b+c-d+e-f) \cdot \begin{vmatrix} a-e & b-d \\ f-d & a-c \end{vmatrix}^2.$$

* In P to find the variables we run our eye along the first row and down the last column, in P′ up the first column and along the first row. If the order be the nth P and P′ are connected by the sign-factor $(-)^{\frac{1}{2}n(n-1)}$.

The general theorem is: *If the first row of a semicirculant be* 1, 2, ..., $2m$, *the* $(m+1)$th *row be* $2m$, $2m-1$, ..., 1, *and the* $m-1$ *rows following each of these be obtained from them by cyclical change, right-handed in the first case and left-handed in the other, the semicirculant is equal to*

$$(-)^{\frac{1}{2}m(m-1)}(1+2+3+4+\ldots)(1-2+3-4+\ldots)\Delta^2 \qquad \text{(IV.)}$$

where Δ *is the determinant of the difference of the matrices of the two persymmetric determinants*

$$P'(m+3, m+4, \ldots, 2m, 1, 2, \ldots, m-1),$$
$$P(2m-1, 2m-2, \ldots, m+1, m, \ldots, 3).$$

A very curious special case is due to Mr. A. M. Nesbitt,* viz., the case where the elements are the even-placed coefficients in the expansion of $(a+b)^{2m+1}$ followed by $m-1$ zeros: the determinant is then equal to $2^{m.(2m+1)}$. For example, when $m=2$ we have

$$\begin{vmatrix} 5 & 10 & 1 & . \\ . & 5 & 10 & 1 \\ . & 1 & 10 & 5 \\ 1 & 10 & 5 & . \end{vmatrix} = 2^{10}.$$

From this is obtainable the equally curious result that *the determinant of the difference of the two persymmetric matrices*

$$\begin{matrix} \binom{2m+1}{1} & \binom{2m+1}{3} & \binom{2m+1}{5} & \cdots\cdots & \binom{2m+1}{2m-3} \\ . & \binom{2m+1}{1} & \binom{2m+1}{3} & \cdots\cdots & \binom{2m+1}{2m-5} \\ . & . & \binom{2m+1}{1} & \cdots\cdots & \binom{2m+1}{2m-7} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ . & . & . & \cdots\cdots & \binom{2m+1}{1} \end{matrix}$$

and

$$\begin{matrix} . & \cdots\cdots & . & . & \binom{2m+1}{0} \\ . & \cdots\cdots & . & \binom{2m+1}{0} & \binom{2m+1}{2} \\ . & \cdots\cdots & \binom{2m+1}{0} & \binom{2m+1}{2} & \binom{2m+1}{4} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \binom{2m+1}{0} & \cdots\cdots & \binom{2m+1}{2m-8} & \binom{2m+1}{2m-6} & \binom{2m+1}{2m-4} \end{matrix}$$

is equal to $2^{m(m-1)}$. (V.)

* See *Educ. Times* (1904, Oct.), lvii. p. 419.

For example, when $m=4$ we have

$$\begin{vmatrix} 9-0 & 84-\ 0 & 126-\ \ 1 \\ 0-0 & 9-\ 1 & 84-\ 36 \\ 0-1 & 0-36 & 9-126 \end{vmatrix} = 2^{12}.$$

8. If the two originating circulants be of odd order, we may approximate to a semicirculant by taking one row more from the one than from the other. In this case also the new determinant is resolvable into factors, but now there is no factor repeated. Thus, taking the five-line instance we have

$$\begin{vmatrix} a & b & c & d & e \\ e & a & b & c & d \\ d & e & a & b & c \\ e & d & c & b & a \\ a & e & d & c & b \end{vmatrix} = \begin{vmatrix} a & b & c & d & e \\ e & a & b & c & d \\ d & e & a & b & c \\ . & d-a & c-b & b-c & a-d \\ . & e-b & d-c & c-d & b-e \end{vmatrix},$$

$$= \begin{vmatrix} a & b+e & c+d & d & e \\ e & a+d & b+c & c & d \\ d & e+c & a+b & b & e \\ . & . & . & b-c & a-d \\ . & . & . & c-d & b-e \end{vmatrix},$$

$$= (a+b+c+d+e \begin{vmatrix} a-e & b-d \\ e-d & a-c \end{vmatrix} \begin{vmatrix} b-c & a-d \\ c-d & b-e \end{vmatrix}.$$

It must be noted, however, that the process here followed is not applicable to the other instances without modifications, and that consequently we are not readily led by it to the formulation of the general result. The analogous difficulty in the case of even-ordered determinants was got over, as we have seen, by a preparatory process which brought about centrosymmetry: here this is impossible. The following theorem suggests a way out.

9. *In every odd-ordered semicirculant there is one column which is centrosymmetric (that is, reversible without change), and there is another, which, if deprived of its first element, has the same property.* (VI.)

If the order-number be $4m+1$, the middle row, being the $(2m+1)$th row of the first originating circulant, is

$$2m+2,\ 2m+3,\ \ldots,\ 4m+1,\ 1,\ 2,\ \ldots,\ 2m+1\,;$$

and the $(2m+2)$th, being the first row of the second originating circulant, is

$$4m+1,\ 4m,\ \ldots,\ 2m+2,\ 2m+1,\ 2m,\ \ldots,\ 2,\ 1.$$

Now, as we have here $2m$ consecutive integers arranged in ascending order, and placed under them the same integers arranged in the reverse order, it follows that the two middle elements of the upper group have below them the same two middle elements in the opposite order: also as we have the first $2m+1$ integers arranged in ascending order, and placed under them the same integers arranged in the reverse order, it follows that the one middle element stands over the same middle element. This amounts to saying that the two rows must be partially represented thus—

$$\begin{array}{l} \ldots\ldots\ldots,\ 3m+1,\ 3m+2,\ \ldots\ldots\ldots,\ m+1,\ \ldots\ldots\ldots \\ \ldots\ldots\ldots,\ 3m+2,\ 3m+1,\ \ldots\ldots\ldots,\ m+1,\ \ldots\ldots\ldots \end{array}$$

Now by reason of the cyclical change from row to row, the elements immediately above $3m+1$ in the first of these two rows (that is, the middle row of the determinant) must be $3m+2$, $3m+3$, ...; and as the elements below are the same, the column in which these elements stand, viz., the mth, must be centrosymmetric. Similar reasoning makes clear that the column having $m+1$, $m+1$ for consecutive elements, viz., the $(3m+1)$th, is also centrosymmetric when deprived of its top element.

If the order-number be $4m+3$, the middle row and the row immediately following it are

$$\begin{array}{cccccccc} 2m+3, & 2m+4, & \ldots, & 4m+3, & 1, & 2, & \ldots, & 2m+1, & 2m+2 \\ 4m+3, & 4m+2, & \ldots, & 2m+3, & 2m+2, & 2m+1, & \ldots, & 2, & 1\ ; \end{array}$$

and these, if we only write the parts which are pertinent to the argument, are

$$\begin{array}{l} \ldots\ldots\ldots,\ 3m+3,\ \ldots\ldots\ldots,\ m+1,\ m+2,\ \ldots\ldots\ldots \\ \ldots\ldots\ldots,\ 3m+3,\ \ldots\ldots\ldots,\ m+2,\ m+1,\ \ldots\ldots\ldots \end{array}$$

so that it is now the $(3m+2)$th column which is perfectly centrosymmetric, and the $(m+1)$th which is approximately so.

10. If the perfectly centrosymmetric column thus proved to exist in a semicirculant of order $2m+1$ be advanced by cyclical change of columns to the first place, the operations performed in § 8 can be followed in exact detail. These are

$$\text{row}_{2m+1} - \text{row}_1, \quad \text{row}_{2m} - \text{row}_2, \quad \ldots\ldots, \quad \text{row}_{m+2} - \text{row}_m$$

and then

$$\text{col}_2 + \text{col}_{2m+1}, \quad \text{col}_3 + \text{col}_{2m}, \quad \ldots\ldots\ldots\ldots\ldots\ldots ;$$

and the value found for the determinant is

$$\sigma . | \, M_1 - M_3 \, | . | \, M_2 - M_3 \, | , \qquad \text{(VII.)}$$

where σ is the sum of the elements, and the M's are matrices formed from the first m rows of the semicirculant, viz., M_1 consisting of the m consecutive columns beginning with the first, M_2 of the m consecutive columns beginning with the second, and M_3 of the last m columns reversed.

For example, in the semicirculant of the 9th order, it being the second column which is centrosymmetric, the first column is moved to the end, whereupon the first four rows stand thus—

$$\begin{matrix} 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 1 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 9 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 8 & 9 & 1 & 2 & 3 & 4 & 5 & 6 & 7, \end{matrix}$$

and the factors of the circulant are seen to be

$$(1+2+\ldots+9)\begin{vmatrix} 2-1 & 3-9 & 4-8 & 5-7 \\ 1-9 & 2-8 & 3-7 & 4-6 \\ 9-8 & 1-7 & 2-6 & 3-5 \\ 8-7 & 9-6 & 1-5 & 2-4 \end{vmatrix} . \begin{vmatrix} 3-1 & 4-9 & 5-8 & 6-7 \\ 2-9 & 3-8 & 4-7 & 5-6 \\ 1-8 & 2-7 & 3-6 & 4-5 \\ 9-7 & 1-6 & 2-5 & 3-4 \end{vmatrix}.$$

11. From this we naturally pass to the semicirculants of odd order in which the two cyclical changes are effected in different directions. It is not necessary, however, to continue to give full details ; the results alone will suffice.

First of all there is the theorem corresponding to that of § 9, viz., *In every odd-ordered semicirculant with opposite cyclical movements the middle column has the first element in the middle place, this being led up to by, the* $(m+1)$th, mth, $(m-1)$th, ..., *3rd, 2nd elements and followed by the same : and the first column has the first element in the first place, this being followed by the* $(2m+1)$th, $2m$th, ..., $(m+2)$th *elements and these repeated.* (VIII.)

This suggests the opening set of operations

$$\text{row}_{2m+1} - \text{row}_{m+1}, \quad \text{row}_{2m} - \text{row}_{m}, \quad \text{row}_{2m-1} - \text{row}_{m-1}, \quad \ldots,$$

after which we are led as before to the required value, viz.,

$$(-)^{\frac{1}{2}m(m-1)}\,\sigma . |\ M_1 - M_2\ | . |\ M_1 - M_3\ | \qquad \text{(IX.)}$$

where σ is the sum of the elements, and the M's are matrices formed from the first m rows, viz., M_1 consisting of the m consecutive columns beginning with the first, M_2 of the m consecutive columns beginning with the second, and M_3 of the last m columns in reversed order.

12. The foregoing investigations suggest others of a more general

character. For example, returning to circulants of the 4th order, we may form a determinant by taking *any* two rows from $C(a, b, c, d)$ and *any* two from $C(d, c, b, a)$. In every such case it is clear that $a+b+c+d$, $a-b+c-d$ would be factors as before; but it is not so readily seen what the co-factor would be. Investigation shows that it takes one of five forms, viz., 0, $(a-c)^2$, $(a-c)(b-d)$, $(b-d)^2$, $(a+b-c-d)(a-b-c+d)$: so that we have the general theorem: *Every determinant formed by taking two rows from* $C(a, b, c, d)$ *and two rows from* $C(d, c, b, a)$ *is resolvable into linear factors or vanishes.* (X.)

Similarly we find that *Every determinant formed by taking any three consecutive rows of* $C(a, b, c, d, e)$ *and any two consecutive rows of* $C(e, d, c, b, a)$ *is resolvable into three factors, one linear and two quadratic.* (XI.)